A COHOMOLOGICAL PROPERTY OF LAGRANGE MULTIPLIERS

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Abstract. The method of Lagrange multipliers relates the critical points of a given function f to the critical points of an auxiliary function F. We establish a cohomological relationship between f and F and use it, in conjunction with the Eagon-Northcott complex, to compute the sum of the Milnor numbers of the critical points in certain situations.

1. Introduction

Let U be an open subset of $\mathbf{R}^n$ and let $f, f_1, \dots, f_r : U \to \mathbf{R}$ be continuously differentiable functions on U. Let $Y \subseteq U$ be the subset defined by $f_1 = \cdots = f_r = 0$ and suppose that the matrix $(\partial f_i/\partial x_j)_{\substack{i=1,\dots,r\\ j=1,\dots,n}}$ has rank r at every point of Y. The usual theorem of Lagrange multipliers says that $\mathbf{a} = (a_1, \dots, a_n) \in Y$ is a critical point of $f|_Y$ if and only if there exists $\mathbf{b} = (b_1, \dots, b_r) \in \mathbf{R}^r$ such that $(\mathbf{a}; \mathbf{b}) \in U \times \mathbf{R}^r$ is a critical point of the auxiliary function $F = f + \sum_{i=1}^r y_i f_i : U \times \mathbf{R}^r \to \mathbf{R}$. The point $\mathbf{b}$ is unique when it exists.

We establish a closer relation between f and F for algebraic varieties over an arbitrary field K. Let $X = \operatorname{Spec}(A)$ be a smooth affine K-scheme of finite type, purely of dimension n, let $f, f_1, \dots, f_r \in A$ and put $I = (f_1, \dots, f_r) \subseteq A$. Put $B = A/I$ and let $Y = \operatorname{Spec}(B)$, a closed subscheme of X. We assume that Y is a smooth K-scheme, purely of codimension r in X. We write $\bar{f}$ for the image of $f \in A$ under the natural map $A \to B$. Let $y_1, \dots, y_r$ be indeterminates and consider $X \times_K \mathbf{A}^r = \operatorname{Spec}(A[y_1, \dots, y_r])$. We shall write $A[y]$ for $A[y_1, \dots, y_r]$. Put $F = f + \sum_{i=1}^r y_i f_i \in A[y]$. Let $\Omega^k_{B/K}$ (resp. $\Omega^k_{A[y]/K}$) be the module of differential k-forms of B (resp. $A[y]$) over K. Let $d_{B/K}\bar{f} \in \Omega^1_{B/K}$ and $d_{A[y]/K}F \in \Omega^1_{A[y]/K}$ be the exterior derivatives of $\bar{f}$ and F, respectively. We consider the complexes $(\Omega^\cdot_{B/K}, \phi_{\bar{f}})$ and $(\Omega^\cdot_{A[y]/K}, \phi_F)$, where $\phi_{\bar{f}} : \Omega^k_{B/K} \to \Omega^{k+1}_{B/K}$ is the map defined by

$$\phi_{\bar{f}}(\omega) = d_{B/K}\bar{f} \wedge \omega$$

and $\phi_F : \Omega^k_{A[y]/K} \to \Omega^{k+1}_{A[y]/K}$ is the map defined by

$$\phi_F(\omega) = d_{A[y]/K}F \wedge \omega.$$

The cohomology of these complexes is supported on the sets of critical points of $\bar{f}$ and F, respectively. The purpose of this note is to prove the following result.

Theorem 1.1. *With the above notation and hypotheses, there are A-module isomorphisms for all i*

$$H^i(\Omega^\cdot_{B/K}, \phi_{\bar{f}}) \simeq H^{i+2r}(\Omega^\cdot_{A[y]/K}, \phi_F).$$

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The cohomology group $H^{n-r}(\Omega^{\cdot}_{B/K},\phi_{\bar{f}})$ plays an important role. For example, if $\bar{f}$ has only isolated critical points on Y, then $H^{n-r}(\Omega^{\cdot}_{B/K},\phi_{\bar{f}})$ is a finite-dimensional K-vector space whose dimension equals the sum of the Milnor numbers of the critical points of $\bar{f}$ on Y. In this case, $H^i(\Omega^{\cdot}_{B/K},\phi_{\bar{f}})=0$ for all $i\neq n-r$. To see this, since the assertion is local, we may assume that $\Omega^1_{B/K}$ is a free B-module of rank $n-r$. We may then choose derivations $D_1,\dots,D_{n-r}\in \mathrm{Der}_K(B)$ that form a basis for $\mathrm{Der}_K(B)$ as B-module and identify $(\Omega^{\cdot}_{B/K},\phi_{\bar{f}})$ with the cohomological Koszul complex on B defined by $D_1\bar{f},\dots,D_{n-r}\bar{f}$. In particular,

$$H^{n-r}(\Omega^{\cdot}_{B/K},\phi_{\bar{f}})\simeq B/(D_1\bar{f},\dots,D_{n-r}\bar{f}).$$

Since this is assumed to be finite-dimensional, the ideal $(D_1\bar{f},\dots,D_{n-r}\bar{f})$ of B has height $n-r$, therefore has depth $n-r$ as well ($\mathrm{Spec}(B)$ smooth implies in particular that B is Cohen-Macaulay). The depth sensitivity of the Koszul complex [5, Theorem 16.8 and Corollary] then implies that all its cohomology in degree $<n-r$ vanishes (and that $D_1\bar{f},\dots,D_{n-r}\bar{f}$ form a regular sequence in B).

Let $\Omega^k_{A/K}$ be the module of differential k-forms of A over K and let $d_{A/K}:\Omega^k_{A/K}\to\Omega^{k+1}_{A/K}$ be exterior differentiation. As a corollary of the proof of Theorem 1.1, we shall obtain the following.

Theorem 1.2. *Under the hypothesis of Theorem* 1.1*, there is an isomorphism of B-modules*

$$H^{n-r}(\Omega^{\cdot}_{B/K},\phi_{\bar{f}})\simeq\left(\Omega^n_{A/K}\Big/(d_{A/K}f\wedge d_{A/K}f_1\wedge\cdots\wedge d_{A/K}f_r\wedge\Omega^{n-r-1}_{A/K})\right)\bigotimes_A B.$$

We write out this isomorphism in a special case. Let $X=\mathbf{A}^n$, so that A is the polynomial ring $K[x_1,\dots,x_n]$ and $f,f_1,\dots,f_r\in K[x_1,\dots,x_n]$. Let J be the ideal of $K[x_1,\dots,x_n]$ generated by the $(r+1)\times(r+1)$-minors of the matrix

$$\begin{bmatrix}\partial f_1/\partial x_1 & \dots & \partial f_1/\partial x_n\\ \dots & \dots & \dots\\ \partial f_r/\partial x_1 & \dots & \partial f_r/\partial x_n\\ \partial f/\partial x_1 & \dots & \partial f/\partial x_n\end{bmatrix}. \tag{1.3}$$

Corollary 1.4. *If $f_1,\dots,f_r\in K[x_1,\dots,x_n]$ define a smooth complete intersection in $\mathbf{A}^n$, then*

$$H^{n-r}(\Omega^{\cdot}_{B/K},\phi_{\bar{f}})\simeq K[x_1,\dots,x_n]/(I+J).$$

Again assume $f,f_1,\dots,f_r\in K[x_1,\dots,x_n]$. Let $d_i=\deg f_i$ for $i=1,\dots,r$, $d_{r+1}=\deg f$, and denote by $f_i^{(d_i)}$ (resp. $f^{(d_{r+1})}$) the homogeneous part of f_i (resp. of f) of highest degree. For $i=1,\dots,r$, Let $\tilde{f}_i$ be the homogenization of f_i, i. e.,

$$\tilde{f}_i=x_0^{d_i}f_i(x_1/x_0,\dots,x_n/x_0)\in K[x_0,\dots,x_n].$$

Using Corollary 1.4 and the Eagon-Northcott complex[2], we shall prove the following result, which was suggested by a theorem of Katz[4, Théorème 5.4.1].

Proposition 1.5. *Suppose that $\tilde{f}_1=\cdots=\tilde{f}_r=0$ defines a smooth complete intersection in $\mathbf{P}^n$ that intersects the hyperplane $x_0=0$ transversally and that $f_1^{(d_1)}=\cdots=f_r^{(d_r)}=f^{(d_{r+1})}=0$ defines a smooth complete intersection in $\mathbf{P}^{n-1}$. If $\mathrm{char}(K)>0$, we assume also that $(d_{r+1},\mathrm{char}(K))=1$. Then $\bar{f}$ has only isolated critical points on the variety $Y\subseteq\mathbf{A}^n$ defined by $f_1=\cdots=f_r=0$ and*

$\dim_K H^{n-r}(\Omega^{\cdot}_{B/K}, \phi_{\bar{f}})$ equals the coefficient of t^{n-r} in the power series expansion at $t = 0$ of the rational function

$$\frac{d_1 \cdots d_r (1-t)^n}{\prod_{i=1}^{r+1}(1-d_i t)}.$$

2. An intermediate complex

We reduce Theorem 1.1 to a related statement. Keeping our hypotheses on X and Y, we shall express the complex $(\Omega^{\cdot}_{A[y]/K}, \phi_F)$ as the total complex associated to a certain double complex and show that the vertical cohomology of this double complex vanishes except in degree r. The intermediate complex referred to in the title of this section will be the horizontal complex associated to this single nonvanishing vertical cohomology group.

We write $K[y]$ for $K[y_1, \dots, y_r]$. Let

$$d_1 : \Omega^k_{A[y]/K[y]} \to \Omega^{k+1}_{A[y]/K[y]}$$

and

$$d_2 : \Omega^k_{A[y]/A} \to \Omega^{k+1}_{A[y]/A}$$

be the exterior derivatives. For $p, q \geq 0$ put

$$C^{p,q} = \Omega^p_{A[y]/K[y]} \bigotimes_{A[y]} \Omega^q_{A[y]/A}.$$

We have

$$\Omega^k_{A[y]/K} \simeq \bigoplus_{p+q=k} C^{p,q}. \tag{2.1}$$

Define $\delta_1 : C^{p,q} \to C^{p+1,q}$ and $\delta_2 : C^{p,q} \to C^{p,q+1}$ by

$$\begin{aligned}
\delta_1(\omega \otimes \omega') &= (d_1 F \wedge \omega) \otimes \omega' \\
\delta_2(\omega \otimes \omega') &= (-1)^p (\omega \otimes (d_2 F \wedge \omega')).
\end{aligned}$$

It is straightforward to check from (2.1) and these definitions that $(\Omega^{\cdot}_{A[y]/K}, \phi_F)$ is the total complex of the double complex $\{C^{p,q}\}$.

Let $\phi'_F : \Omega^k_{A[y]/A} \to \Omega^{k+1}_{A[y]/A}$ be defined by

$$\phi'_F(\omega') = d_2 F \wedge \omega'.$$

In coordinate form, we have

$$d_2 F = \sum_{i=1}^r \frac{\partial F}{\partial y_i}\, dy_i = \sum_{i=1}^r f_i\, dy_i,$$

so the complex $(\Omega^{\cdot}_{A[y]/A}, \phi'_F)$ is isomorphic to the (cohomological) Koszul complex on $A[y]$ defined by $f_1, \dots, f_r$. This Koszul complex decomposes into a direct sum of copies (indexed by the monomials in $y_1, \dots, y_r$) of the Koszul complex on A defined by $f_1, \dots, f_r$. We denote this latter Koszul complex by $\mathrm{Kos}(A; f_1, \dots, f_r)$. Our hypothesis that Y is a smooth complete intersection defined by $f_1, \dots, f_r$ implies that $f_1, \dots, f_r$ form a regular sequence in the local ring of X at any point of Y. This gives the following result.

Lemma 2.2. *The cohomology of the complex* $\mathrm{Kos}(A; f_1, \dots, f_r)$ *is given by*

$$H^i(\mathrm{Kos}(A; f_1, \dots, f_r)) = 0 \quad \textit{if } i \neq r,$$
$$H^r(\mathrm{Kos}(A; f_1, \dots, f_r)) = B.$$

In view of the remarks preceding the lemma, we get the following.

Corollary 2.3. *The cohomology of the complex* $(\Omega^{\cdot}_{A[y]/A}, \phi'_F)$ *is given by*

$$H^i(\Omega^{\cdot}_{A[y]/A}, \phi'_F) = 0 \quad \textit{if } i \neq r, \tag{2.4}$$
$$H^r(\Omega^{\cdot}_{A[y]/A}, \phi'_F) = B[y_1, \dots, y_r]. \tag{2.5}$$

Equation (2.4) says that the vertical cohomology of the double complex $\{C^{p,q}\}$ vanishes except in degree r. Equation (2.5) and standard results in homological algebra relating the vertical cohomology of a double complex to the cohomology of its associated total complex then imply that

$$H^{i+r}(\Omega^{\cdot}_{A[y]/K}, \phi_F) \simeq H^i\left(\Omega^{\cdot}_{A[y]/K[y]} \bigotimes_{A[y]} B[y], \bar{\delta}_1\right) \tag{2.6}$$

for all i, where $\bar{\delta}_1$ is the map induced by δ_1.

We have isomorphisms

$$\Omega^k_{A[y]/K[y]} \bigotimes_{A[y]} B[y] \simeq \Omega^k_{A/K} \bigotimes_A B[y] \tag{2.7}$$

for all k. Let $d_{A/K} : \Omega^k_{A/K} \to \Omega^{k+1}_{A/K}$ be the exterior derivative. By abuse of notation, we denote by $d_{A/K}F$ the element

$$d_{A/K}F = d_{A/K}f \otimes 1 + \sum_{i=1}^r d_{A/K}f_i \otimes y_i \in \Omega^1_{A/K} \bigotimes_A B[y]. \tag{2.8}$$

There is a canonical map

$$\left(\Omega^k_{A/K} \bigotimes_A B[y]\right) \bigotimes_{B[y]} \left(\Omega^l_{A/K} \bigotimes_A B[y]\right) \to \Omega^{k+l}_{A/K} \bigotimes_A B[y]$$

that sends $(\omega_1 \otimes \alpha_1) \otimes (\omega_2 \otimes \alpha_2)$ to $(\omega_1 \wedge \omega_2) \otimes (\alpha_1\alpha_2)$. We denote by

$$(\omega_1 \otimes \alpha_1) \wedge (\omega_2 \otimes \alpha_2)$$

the image of $(\omega_1 \otimes \alpha_1) \otimes (\omega_2 \otimes \alpha_2)$ under this map. Define $\tilde{\phi}_F : \Omega^k_{A/K} \bigotimes_A B[y] \to \Omega^{k+1}_{A/K} \bigotimes_A B[y]$ by

$$\tilde{\phi}_F(\omega \otimes \alpha) = d_{A/K}F \wedge (\omega \otimes \alpha).$$

It is straightforward to check that under the isomorphism (2.7), the map $\bar{\delta}_1$ on the left-hand side is identified with the map $\tilde{\phi}_F$ on the right-hand side. Thus (2.6) gives isomorphisms for all i

$$H^{i+r}(\Omega^{\cdot}_{A[y]/K}, \phi_F) \simeq H^i(\Omega^{\cdot}_{A/K} \bigotimes_A B[y], \tilde{\phi}_F). \tag{2.9}$$

The main effort in this paper will be devoted to proving the following.

Theorem 2.10. *There is an injective quasi-isomorphism of complexes of A-modules*

$$(\Omega^{\cdot}_{B/K}[-r], \phi_{\bar{f}}) \hookrightarrow (\Omega^{\cdot}_{A/K} \bigotimes_A B[y], \tilde{\phi}_F),$$

where $\Omega^{\cdot}_{B/K}[-r]$ is the complex with $\Omega^{i}_{B/K}[-r] = \Omega^{i-r}_{B/K}$. In particular, there are isomorphisms for all i

$$H^{i-r}(\Omega^{\cdot}_{B/K}, \phi_{\bar{f}}) \simeq H^{i}(\Omega^{\cdot}_{A/K} \bigotimes_A B[y], \tilde{\phi}_F).$$

Theorem 1.1 clearly follows from Theorem 2.10 and equation (2.9). The proof of Theorem 2.10 will be carried out in sections 3 and 4. In section 3, we define an isomorphism of $(\Omega^{\cdot}_{B/K}[-r], \phi_{\bar{f}})$ with a subcomplex $L^{\cdot}$ of $(\Omega^{\cdot}_{A/K} \bigotimes_A B[y], \tilde{\phi}_F)$. In section 4, we show that the inclusion of $L^{\cdot}$ into $\Omega^{\cdot}_{A/K} \bigotimes_A B[y]$ is a quasi-isomorphism.

3. Proof of Theorem 2.10

We begin by using the degree in the y-variables to construct an increasing filtration on the complex $(\Omega^{\cdot}_{A/K} \bigotimes_A B[y], \tilde{\phi}_F)$. Let $F_d B[y]$ be the B-module of all polynomials of degree $\leq d$ in $y_1, \ldots, y_r$. We define the filtration $F.$ on $\Omega^{\cdot}_{A/K} \bigotimes_A B[y]$ by setting

$$F_d(\Omega^{k}_{A/K} \bigotimes_A B[y]) = \Omega^{k}_{A/K} \bigotimes_A F_{d+k} B[y].$$

By (2.8) we see that

$$\tilde{\phi}_F\left(F_d(\Omega^{k}_{A/K} \bigotimes_A B[y]) \right) \subseteq F_d(\Omega^{k+1}_{A/K} \bigotimes_A B[y]),$$

hence we have a filtered complex. Since $F_d B[y] = 0$ for $d < 0$, we have

$$F_d(\Omega^{k}_{A/K} \bigotimes_A B[y]) = 0 \quad \text{for } d < -k.$$

Furthermore, $F_0 B[y] = B$, so we make the identification

$$F_{-k}(\Omega^{k}_{A/K} \bigotimes_A B[y]) = \Omega^{k}_{A/K} \bigotimes_A B. \tag{3.1}$$

We define a map

$$\Phi : \Omega^{k}_{A/K} \bigotimes_A B \to \Omega^{k+r}_{A/K} \bigotimes_A B$$

by the formula

$$\Phi(\xi) = (-1)^{kr} (d_{A/K} f_1 \otimes 1) \wedge \cdots \wedge (d_{A/K} f_r \otimes 1) \wedge \xi$$

for $\xi \in \Omega^{k}_{A/K} \bigotimes_A B$.

Lemma 3.2. $\ker \Phi = \sum_{i=1}^{r} (d_{A/K} f_i \otimes 1) \wedge (\Omega^{k-1}_{A/K} \bigotimes_A B)$.

Proof. It suffices to check equality locally, so we may assume that $\Omega^{1}_{A/K}$ is a free A-module, of rank n. Then $\Omega^{1}_{A/K} \bigotimes_A B$ is a free B-module of rank n and

$$\Omega^{k}_{A/K} \bigotimes_A B \simeq \bigwedge^{k} (\Omega^{1}_{A/K} \bigotimes_A B).$$

We are thus in the situation of Saito[6]. The smooth complete intersection hypothesis implies that the ideal of B generated by the coefficients of $(d_{A/K}f_1\otimes 1)\wedge\cdots\wedge(d_{A/K}f_r\otimes 1)$ relative to the basis of $\Omega^r_{A/K}\bigotimes_A B$ obtained by taking r-fold exterior products of a basis of $\Omega^1_{A/K}\bigotimes_A B$ is the unit ideal. The desired conclusion then follows from part (i) of the theorem of [6].

It follows from Lemma 3.2 that

$$(\Omega^k_{A/K}\bigotimes_A B)/\ker\Phi \simeq \left(\Omega^k_{A/K}\Big/\sum_{i=1}^r (d_{A/K}f_i\wedge\Omega^{k-1}_{A/K})\right)\bigotimes_A B.$$

By a standard result, this is just $\Omega^k_{B/K}$. Using the identification (3.1) (with k replaced by $k+r$), we see that Φ induces an imbedding

$$\bar{\Phi}:\Omega^k_{B/K}\hookrightarrow \Omega^{k+r}_{A/K}\bigotimes_A B[y]$$

with image in $F_{-k-r}(\Omega^{k+r}_{A/K}\bigotimes_A B[y])$. Equation (2.8) implies that

$$\tilde{\phi}_F(\Phi(\xi)) = (d_{A/K}f\otimes 1)\wedge\Phi(\xi) \tag{3.3}$$

for $\xi\in\Omega^k_{A/K}\bigotimes_A B$, from which it is easily seen that $\bar{\Phi}$ is a map of complexes.

Equation (3.3) gives additional information. Define a subcomplex $(L^\cdot,\tilde{\phi}_F)$ of $(\Omega^\cdot_{A/K}\bigotimes_A B[y],\tilde{\phi}_F)$ by setting

$$L^k=\{\xi\in F_{-k}(\Omega^k_{A/K}\bigotimes_A B[y])\mid \tilde{\phi}_F(\xi)\in F_{-k-1}(\Omega^{k+1}_{A/K}\bigotimes_A B[y])\}.$$

Proposition 3.4. *The map $\bar{\Phi}$ is an isomorphism of complexes from $(\Omega^\cdot_{B/K}[-r],\phi_{\bar{f}})$ onto $(L^\cdot,\tilde{\phi}_F)$.*

Proof. Let $\omega\in\Omega^{k-r}_{B/K}$. Then clearly $\bar{\Phi}(\omega)\in F_{-k}(\Omega^k_{A/K}\bigotimes_A B[y])$ and, by (3.3), $\bar{\Phi}(\omega)\in L^k$. Thus $\bar{\Phi}$ is an injective homomorphism of complexes whose image is contained in $L^\cdot$. It only remains to prove that $L^k\subseteq\bar{\Phi}(\Omega^{k-r}_{B/K})$ for all k.

Let $\xi\in F_{-k}(\Omega^k_{A/K}\bigotimes_A B[y])$. We may write

$$\xi=\sum_j \omega_j\otimes b_j$$

with $\omega_j\in\Omega^k_{A/K}$ and $b_j\in B$. We have

$$d_{A/K}F\wedge\xi=\sum_j (d_{A/K}f\wedge\omega_j)\otimes b_j+\sum_{i=1}^r\left(\sum_j (d_{A/K}f_i\wedge\omega_j)\otimes b_jy_i\right).$$

Since $\Omega^{k+1}_{A/K}\bigotimes_A B[y]$ is locally a free $B[y]$-module, $\tilde{\phi}_F(\xi)\in F_{-k-1}(\Omega^{k+1}_{A/K}\bigotimes_A B[y])$ if and only if

$$\sum_j (d_{A/K}f_i\wedge\omega_j)\otimes b_j=0\quad\text{for } i=1,\dots,r,$$

i. e., $\xi\in L^k$ if and only if $(d_{A/K}f_i\otimes 1)\wedge\xi=0$ for $i=1,\dots,r$. Thus the proof will be completed by the following result.

Lemma 3.5. *Suppose* $\xi \in \Omega^k_{A/K} \bigotimes_A B$ *satisfies*

$$(d_{A/K} f_i \otimes 1) \wedge \xi = 0 \quad \textit{for } i = 1, \dots, r.$$

Then $\xi \in \operatorname{im} \Phi$.

Proof. It suffices to check the condition locally, i. e., to show that for any maximal ideal $\bar{\mathbf{m}}$ of B, if

$$(d_{A/K} f_i \otimes 1)_{\bar{\mathbf{m}}} \wedge \xi_{\bar{\mathbf{m}}} = 0 \quad \text{for } i = 1, \dots, r, \tag{3.6}$$

then $\xi_{\bar{\mathbf{m}}} \in \operatorname{im} \Phi_{\bar{\mathbf{m}}}$. Let $\mathbf{m}$ be the maximal ideal of A corresponding to the maximal ideal $\bar{\mathbf{m}}$ of B. The smooth complete intersection hypotheses implies that $(d_{A/K} f_1)_{\mathbf{m}}, \dots, (d_{A/K} f_r)_{\mathbf{m}}$ can be extended to a basis of $(\Omega^1_{A/K})_{\mathbf{m}}$ as $A_{\mathbf{m}}$-module. This implies that $\{(d_{A/K} f_i \otimes 1)_{\bar{\mathbf{m}}}\}_{i=1}^r$ can be extended to a basis of $(\Omega^1_{A/K} \otimes_A B)_{\bar{\mathbf{m}}}$ as $B_{\bar{\mathbf{m}}}$-module. Since

$$(\Omega^k_{A/K} \otimes_A B)_{\bar{\mathbf{m}}} \simeq \bigwedge^k (\Omega^1_{A/K} \otimes_A B)_{\bar{\mathbf{m}}},$$

our k-form $\xi_{\bar{\mathbf{m}}}$ can be written as a sum of k-fold exterior products of these basis elements multiplied by elements of $B_{\bar{\mathbf{m}}}$. But then (3.6) implies that $(d_{A/K} f_i \otimes 1)_{\bar{\mathbf{m}}}$ must appear in each of these k-fold exterior products, i. e.,

$$\xi_{\bar{\mathbf{m}}} = (d_{A/K} f_1 \otimes 1)_{\bar{\mathbf{m}}} \wedge \cdots \wedge (d_{A/K} f_r \otimes 1)_{\bar{\mathbf{m}}} \wedge \eta_{\bar{\mathbf{m}}}$$

for some $\eta_{\bar{\mathbf{m}}} \in (\Omega^{k-r}_{A/K} \otimes_A B)_{\bar{\mathbf{m}}}$. Hence $\xi_{\bar{\mathbf{m}}} \in \operatorname{im} \Phi_{\bar{\mathbf{m}}}$.

Proof of Theorem 1.2. Proposition 3.4 implies that

$$H^{n-r}(\Omega^{\cdot}_{B/K}, \phi_{\bar{f}}) \simeq L^n / \tilde{\phi}_F(L^{n-1}).$$

But from the definition

$$L^n = \Omega^n_{A/K} \bigotimes_A B$$

and from Proposition 3.4

$$L^{n-1} = \Phi(\Omega^{n-r-1}_{A/K} \bigotimes_A B).$$

Theorem 1.2 now follows from equation (3.3) and the definition of Φ.

4. Completion of the proof

To complete the proof, we show that the inclusion $L^{\cdot} \hookrightarrow \Omega^{\cdot}_{A/K} \bigotimes_A B[y]$ is a quasi-isomorphism. For this, it suffices to show that the corresponding map of associated graded complexes relative to the filtration $F.$ is a quasi-isomorphism. We write gr_d to denote these associated graded complexes.

It is easily seen that

$$\operatorname{gr}_d(\Omega^k_{A/K} \bigotimes_A B[y]) = \Omega^k_{A/K} \bigotimes_A B[y]^{(d+k)},$$

where $B[y]^{(d)}$ denotes the B-module of homogeneous polynomials of degree d in $y_1, \dots, y_r$. Furthermore, the differential $\operatorname{gr}(\tilde{\phi}_F)$ of this associated graded complex is given by

$$\operatorname{gr}(\tilde{\phi}_F)(\xi) = \sum_{i=1}^r (d_{A/K} f_i \otimes y_i) \wedge \xi$$

for $\xi \in \Omega^k_{A/K} \bigotimes_A B[y]^{(d+k)}$. It is also easy to see that $F.$ induces the "stupid" filtration on $L^{\cdot}$, i. e.,

$$F_d L^k = \begin{cases} L^k & \text{if } d \geq -k, \\ 0 & \text{if } d < -k, \end{cases}$$

hence $\mathrm{gr}_d(L^{\cdot})$ is the complex with L^{-d} in degree $-d$ and zeros elsewhere if $-n \leq d \leq 0$ and is the zero complex otherwise. Thus the assertion that

$$\mathrm{gr}_d(L^{\cdot}) \hookrightarrow \mathrm{gr}_d(\Omega^{\cdot}_{A/K} \bigotimes_A B[y])$$

is a quasi-isomorphism is equivalent to the assertion that

$$0 \to L^{-d} \to \Omega^{-d}_{A/K} \bigotimes_A B[y]^{(0)} \to \Omega^{1-d}_{A/K} \bigotimes_A B[y]^{(1)} \to \cdots \to \Omega^n_{A/K} \bigotimes_A B[y]^{(d+n)} \to 0$$

is exact for $-n \leq d \leq 0$ and

$$0 \to \Omega^0_{A/K} \bigotimes_A B[y]^{(d)} \to \cdots \to \Omega^n_{A/K} \bigotimes_A B[y]^{(d+n)} \to 0$$

is exact for $d > 0$. The definition of $L^{\cdot}$ shows that the sequence

$$0 \to L^{-d} \to \Omega^{-d}_{A/K} \bigotimes_A B[y]^{(0)} \to \Omega^{1-d}_{A/K} \bigotimes_A B[y]^{(1)}$$

is exact for $-n \leq d \leq 0$. Thus we need to show that the sequence

$$\Omega^{k-1}_{A/K} \bigotimes_A B[y]^{(d+k-1)} \to \Omega^k_{A/K} \bigotimes_A B[y]^{(d+k)} \to \Omega^{k+1}_{A/K} \bigotimes_A B[y]^{(d+k+1)}$$

is exact whenever $d > -k$.

This can be summarized in the following result.

Proposition 4.1. *$H^k(\Omega^{\cdot}_{A/K} \bigotimes_A B[y]^{(\cdot+d)}, \mathrm{gr}(\tilde{\phi}_F)) = 0$ for $d > -k$.*

Proof. The complex in question is a complex of A-modules supported on $\mathrm{Spec}(B)$. Hence to prove the desired vanishing, we may first localize at a maximal ideal of A containing $f_1, \ldots, f_r$. Thus we may assume that A is a smooth local K-algebra of dimension n whose maximal ideal $\mathbf{m}$ contains $f_1, \ldots, f_r$ and that $B = A/(f_1, \ldots, f_r)$ is a smooth local K-algebra of dimension $n-r$. This implies that there exist $f_{r+1}, \ldots, f_n \in \mathbf{m}$ such that $d_{A/K}f_1, \ldots, d_{A/K}f_n$ form a basis for $\Omega^1_{A/K}$. To simplify notation, we write

$$\Omega_{i_1\cdots i_k} = (d_{A/K}f_{i_1} \otimes 1) \wedge \cdots \wedge (d_{A/K}f_{i_k} \otimes 1) \in \Omega^k_{A/K} \bigotimes_A B[y].$$

Then $\Omega^k_{A/K} \bigotimes_A B[y]$ is a free $B[y]$-module with basis

$$\{\Omega_{i_1\cdots i_k} \mid 1 \leq i_1 < \cdots < i_k \leq n\} \tag{4.2}$$

and is a free B-module with basis

$$\{y_1^{a_1} \cdots y_r^{a_r} \Omega_{i_1\cdots i_k} \mid a_1, \ldots, a_r \in \mathbf{N}, \quad 1 \leq i_1 < \cdots < i_k \leq n\}. \tag{4.3}$$

The differential $\mathrm{gr}(\tilde{\phi}_F)$ of the complex takes the form

$$\mathrm{gr}(\tilde{\phi}_F)(\xi) = \sum_{i=1}^r y_i \Omega_i \wedge \xi.$$

We proceed by induction on r. Suppose $r = 1$ and let $\xi \in \Omega^k_{A/K} \bigotimes_A B[y_1]^{(d+k)}$. The condition $d > -k$ implies that ξ is divisible by y_1, i. e., ξ can be written

$$\xi = \sum_{1 \le i_1 < \cdots < i_k \le n} y_1 \xi(i_1, \ldots, i_k) \Omega_{i_1 \cdots i_k}$$

with $\xi(i_1, \ldots, i_k) \in B[y_1]^{(d+k-1)}$. The condition that ξ be a cocycle is that

$$y_1 \Omega_1 \wedge \xi = 0.$$

Since (4.2) is a basis, we see that this is the case if and only if $\xi(i_1, \ldots, i_k) \neq 0$ implies $i_1 = 1$. Put

$$\eta = \sum_{2 \le i_2 < \cdots < i_k \le n} \xi(1, i_2, \ldots, i_k) \Omega_{i_2 \cdots i_k}.$$

Then $\eta \in \Omega^{k-1}_{A/K} \bigotimes_A B[y_1]^{(d+k-1)}$ and

$$\xi = y_1 \Omega_1 \wedge \eta,$$

so ξ is a coboundary.

Now let $r \ge 2$ and suppose the proposition is true for $r - 1$. Let

$$\xi \in \Omega^k_{A/K} \bigotimes_A B[y]^{(d+k)}$$

be a cocycle, i. e.,

$$\sum_{i=1}^{r} y_i \Omega_i \wedge \xi = 0. \tag{4.4}$$

Let h be the highest power of y_1 appearing in any term of ξ (in the decomposition corresponding to the basis (4.3)) and let $\xi^{(h)}$ be the sum of all terms of degree h in y_1. Suppose $h > 0$. Looking at the terms of degree $h + 1$ in y_1 in equation (4.4) gives

$$\Omega_1 \wedge \xi^{(h)} = 0,$$

hence

$$\xi^{(h)} = \sum_{2 \le i_2 < \cdots < i_k \le n} y_1 \eta(i_2, \ldots, i_k) \Omega_{1 i_2 \cdots i_k}$$

for some $\eta(i_2, \ldots, i_k) \in B[y]^{(d+k-1)}$. Put

$$\eta = \sum_{2 \le i_2 < \cdots < i_k \le n} \eta(i_2, \ldots, i_k) \Omega_{i_2 \cdots i_k}.$$

Then $\eta \in \Omega^{k-1}_{A/K} \bigotimes_A B[y]^{(d+k-1)}$ and the highest power of y_1 appearing in

$$\xi - \sum_{i=1}^{r} y_i \Omega_i \wedge \eta$$

is $\le h - 1$.

We may thus reduce to the case $h = 0$, i. e., y_1 does not appear in ξ. Equation (4.4) then implies the two equalities

$$\Omega_1 \wedge \xi = 0 \tag{4.5}$$

$$\sum_{i=2}^{r} y_i \Omega_i \wedge \xi = 0. \tag{4.6}$$

From (4.5) we have

$$\xi = \sum_{2\leq i_2<\cdots<i_k\leq n} \xi(i_2,\dots,i_k)\Omega_{1i_2\cdots i_k}$$

with $\xi(i_2,\dots,i_k) \in B[y_2,\dots,y_r]^{(d+k)}$. Put

$$\xi' = \sum_{2\leq i_2<\cdots<i_k\leq n} \xi(i_2,\dots,i_k)\Omega_{i_2\cdots i_k} \in \Omega^{k-1}_{A/K} \bigotimes_A B[y_2,\dots,y_r]^{(d+k)}.$$

By (4.6) we have

$$\sum_{i=2}^r y_i\Omega_i \wedge \xi' = 0,$$

i. e., ξ' is a $(k-1)$-cocycle in the complex $(\Omega^{\cdot}_{A/K} \bigotimes_A B[y_2,\dots,y_r]^{(\cdot+d+1)}, \bar{\phi})$, where

$$\bar{\phi}(\zeta) = \sum_{i=2}^r y_i\Omega_i \wedge \zeta.$$

But $d > -k$ implies $d+1 > -(k-1)$, so the induction hypothesis for $r-1$ applies and we conclude that ξ' is a coboundary. This means there exists

$$\eta' \in \Omega^{k-2}_{A/K} \bigotimes_A B[y_2,\dots,y_r]^{(d+k-1)}$$

such that

$$\sum_{i=2}^r y_i\Omega_i \wedge \eta' = \xi'.$$

If we put $\eta = -\Omega_1 \wedge \eta' \in \Omega^{k-1}_{A/K} \bigotimes_A B[y]^{(d+k-1)}$, then

$$\begin{aligned}\sum_{i=1}^r y_i\Omega_i \wedge \eta &= \Omega_1 \wedge \xi' \\ &= \xi,\end{aligned}$$

hence ξ is a coboundary.

5. Proof of Proposition 1.5

Let $f, f_1,\dots,f_r \in K[x_1,\dots,x_n]$ satisfy the hypotheses of Proposition 1.5 and let $Y \subseteq \mathbf{A}^n$ be the variety $f_1 = \cdots = f_r = 0$. We begin by showing that $\bar{f}$ has only isolated critical points on Y. For notational convenience we write $K[x]$ for $K[x_1,\dots,x_n]$. Let $I \subseteq K[x]$ be the ideal generated by $f_1,\dots,f_r$ and let $J \subseteq K[x]$ be the ideal generated by the $(r+1)\times(r+1)$-minors of the matrix (1.3). Put

$$Z = V(I+J) \subseteq \mathbf{A}^n.$$

The underlying point set of Z is the set of critical points of $\bar{f}$ on Y. We wish to show it is finite. If not, then $\dim Z \geq 1$, so $\dim \tilde{Z} \geq 1$ also, where $\tilde{Z}$ denotes the closure of Z in $\mathbf{P}^n$ under the natural compactification by adjoining the hyperplane $x_0 = 0$ at infinity. This would imply that the intersection $\tilde{Z}\cap\{x_0 = 0\}$ is nonempty. We prove that in fact it is empty, therefore Z must be finite.

Consider the matrix

$$\begin{bmatrix} \partial f_1^{(d_1)}/\partial x_1 & \dots & \partial f_1^{(d_1)}/\partial x_n \\ \dots & \dots & \dots \\ \partial f_r^{(d_r)}/\partial x_1 & \dots & \partial f_r^{(d_r)}/\partial x_n \\ \partial f^{(d_{r+1})}/\partial x_1 & \dots & \partial f^{(d_{r+1})}/\partial x_n \end{bmatrix}. \tag{5.1}$$

Let $I' \subseteq K[x]$ be the ideal generated by $f_1^{(d_1)}, \dots, f_r^{(d_r)}$ and let J' be the ideal generated by the $(r+1)\times(r+1)$-minors of the matrix (5.1). For a homogeneous ideal $Q \subseteq K[x]$, we denote by $V(Q) \subseteq \mathbf{A}^n$ the affine variety it defines and by $\tilde{V}(Q) \subseteq \mathbf{P}^{n-1}$ the projective variety it defines. As point sets we have

$$\tilde{Z} \cap \{x_0 = 0\} = \tilde{V}(I' + J').$$

Suppose there exists a point $z \in \tilde{Z} \cap \{x_0 = 0\}$. By hypothesis, $f_1^{(d_1)} = \dots = f_r^{(d_r)} = 0$ defines a smooth complete intersection in $\mathbf{P}^{n-1}$, so at any set of homogeneous coordinates for the point z the first r rows of the matrix (5.1) are linearly independent. But since all $(r+1)\times(r+1)$-minors of (5.1) vanish at any set of homogeneous coordinates for z, the last row must be a linear combination of the first r rows, say,

$$\left(\frac{\partial f^{(d_{r+1})}}{\partial x_1}, \dots, \frac{\partial f^{(d_{r+1})}}{\partial x_n}\right) = \sum_{i=1}^r c_i \left(\frac{\partial f_i^{(d_i)}}{\partial x_1}, \dots, \frac{\partial f_i^{(d_i)}}{\partial x_n}\right)$$

when evaluated at homogeneous coordinates for z, where the c_i lie in the algebraic closure of K. For $j = 1, \dots, n$, we then have

$$x_j \frac{\partial f^{(d_{r+1})}}{\partial x_j} = \sum_{i=1}^r c_i x_j \frac{\partial f_i^{(d_i)}}{\partial x_j}$$

when evaluated at homogeneous coordinates for z. Summing these equations over j and using the Euler relation gives

$$d_{r+1} f^{(d_{r+1})} = \sum_{i=1}^r c_i d_i f_i^{(d_i)}$$

when evaluated at these homogeneous coordinates. Each $f_i^{(d_i)}$ vanishes at z and we are assuming $(d_{r+1}, \mathrm{char}(K)) = 1$ if $\mathrm{char}(K) > 0$, therefore $f^{(d_{r+1})}$ vanishes at z also. But this contradicts the hypothesis that $f^{(d_{r+1})} = f_1^{(d_1)} = \dots = f_r^{(d_r)} = 0$ defines a smooth complete intersection in $\mathbf{P}^{n-1}$.

To calculate $\dim_K H^{n-r}(\Omega^{\cdot}_{B/K}, \phi_{\bar{f}})$, we begin by recalling the definition of the Eagon-Northcott complex[2]. In order to facilitate application of the results of [2, 3], we describe these complexes homologically, rather than cohomologically. For $p = 0, 1, \dots, n-r$, let $C_p^{(1)}$ be the free $K[x]$-module with the following basis. For $p = 0$, $C_0^{(1)} = K[x]$ with basis 1. For $p \geq 1$, $C_p^{(1)}$ has basis

$$\xi_{i_1} \cdots \xi_{i_{p+r}} \eta_1^{j_1} \cdots \eta_{r+1}^{j_{r+1}},$$

where $1 \leq i_1 < \dots < i_{p+r} \leq n$ and $j_1, \dots, j_{r+1}$ are nonnegative integers satisfying

$$j_1 + \dots + j_{r+1} = p - 1.$$

Thus $C_p^{(1)}$ has rank $\binom{n}{p+r}\binom{p+r-1}{r}$. Let $\delta_1 : C_p^{(1)} \to C_{p-1}^{(1)}$ be the $K[x]$-module homomorphism whose action on basis elements is defined as follows. For $p = 1$,

$$\delta_1(\xi_{i_1}\cdots\xi_{i_{r+1}}) = \frac{\partial(f_1,\dots,f_r,f)}{\partial(x_{i_1},\cdots,x_{i_{r+1}})}.$$

For $p > 1$,

$$\delta_1(\xi_{i_1}\cdots\xi_{i_{p+r}}\eta_1^{j_1}\cdots\eta_{r+1}^{j_{r+1}}) = \sum_{\substack{l=1\\ j_l>0}}^{r+1}\sum_{m=1}^{p+r}(-1)^{m-1}\frac{\partial f_l}{\partial x_{i_m}}\xi_{i_1}\cdots\hat{\xi}_{i_m}\cdots\xi_{i_{p+r}}\eta_1^{j_1}\cdots\eta_l^{j_l-1}\cdots\eta_{r+1}^{j_{r+1}}.$$

This is the Eagon-Northcott complex associated to the matrix (1.3).

We define a grading and filtration on $C_{\cdot}^{(1)}$ as follows. Let $K[x]$ have the usual grading and define

$$\text{degree}(\xi_{i_1}\cdots\xi_{i_{p+r}}\eta_1^{j_1}\cdots\eta_{r+1}^{j_{r+1}}) = (j_1+1)d_1+\cdots+(j_{r+1}+1)d_{r+1}-(p+r).$$

This defines a grading on $C_p^{(1)}$ which in turn defines a filtration on $C_p^{(1)}$ by letting level k of the filtration be the K-span of homogeneous elements of degree $\leq k$. It is straighforward to check that δ_1 preserves this filtration. We let $(\bar{C}_{\cdot}^{(1)},\bar{\delta}_1)$ be the associated graded complex. It is easily checked that $(\bar{C}_{\cdot}^{(1)},\bar{\delta}_1)$ is the Eagon-Northcott complex associated to the matrix (5.1).

Proposition 5.2. *For $p > 0$,*

$$H_p(\bar{C}_{\cdot}^{(1)},\bar{\delta}_1) = 0$$

and the Hilbert-Poincaré series of the graded module $H_0(\bar{C}_{\cdot}^{(1)},\bar{\delta}_1)$ has the form

$$\frac{G(t)(1-t)^{n-r}+H(t)(1-t)^{n-r+1}}{(1-t)^n},$$

where $G(t), H(t)$ are polynomials and $G(1)$ equals the coefficient of t^{n-r} in the power series expansion at $t = 0$ of

$$\frac{(1-t)^n}{\prod_{i=1}^{r+1}(1-d_it)}.$$

Proof. We show that the ideal J' has depth $n-r$. The vanishing of $H_p(\bar{C}_{\cdot}^{(1)},\bar{\delta}_1)$ for $p > 0$ then follows from [2, Theorem 1] and the assertion about the Hilbert-Poincaré series of $H_0(\bar{C}_{\cdot}^{(1)},\bar{\delta}_1)$ follows from Theorems 1, 2, and the Lemma in [3]. (Note that, in the notation of [3], we take $\mu_i = d_i$, $\nu_j = -1$, so that the entry in row i, column j of the matrix (5.1) is a homogeneous polynomial of degree $\mu_i + \nu_j = d_i - 1$.) In fact, it is shown in [2] that the depth of J' is $\leq n-r$, so we only need to show that the depth is $\geq n-r$.

Since the depth and the height of J' are equal, it suffices to show that the height of J' is $\geq n-r$. We proved at the beginning of this section that $\tilde{V}(I'+J')$ is empty. But this says that $\tilde{V}(I')$ and $\tilde{V}(J')$ have empty intersection. By hypothesis, $\tilde{V}(I')$ is a smooth complete intersection, purely of dimension $n-1-r$, hence all components of $\tilde{V}(J')$ must have dimension $< r$. This implies that all components of $V(J')$ in $\mathbf{A}^n$ have dimension $\leq r$, i. e., have codimension $\geq n-r$. This is equivalent to the assertion that the height of J' is $\geq n-r$.

Put $B' = K[x]/I'$. We consider the related complex $(\bar{C}^{(1)}_{\cdot} \bigotimes_{K[x]} B', \bar{\delta}_1 \otimes 1)$.

Proposition 5.3. *For $p > 0$,*

$$H_p\left(\bar{C}^{(1)}_{\cdot} \bigotimes_{K[x]} B', \bar{\delta}_1 \otimes 1\right) = 0.$$

Proof. The complex $\bar{C}^{(1)}_{\cdot} \bigotimes_{K[x]} B'$ is the Eagon-Northcott complex of the image of the matrix (5.1) in B'. So by the same argument used in the proof of Proposition 5.2, it suffices to show that the ideal $(J'+I')/I'$ in B' has depth $\geq n-r$. Since $f_1^{(d_1)}, \dots, f_r^{(d_r)}$ is a regular sequence, the ring B' is Cohen-Macaulay. Therefore the height and the depth of $(J'+I')/I'$ are equal, so we are again reduced to showing that the height of $(J'+I')/I'$ is $\geq n-r$. Let $\mathbf{m}$ denote the ideal $(x_1, \dots, x_n)$ of $K[x]$. Since $\tilde{V}(I'+J') = \emptyset$, the only prime ideal of B' containing $(J'+I')/I'$ is $\mathbf{m}/I'$. So it suffices to show that $\mathbf{m}/I'$ has height $\geq n-r$. Let $\mathbf{p}$ be a minimal prime ideal of $K[x]$ containing I'. Since $\tilde{V}(I')$ is purely of codimension r in $\mathbf{P}^{n-1}$, $\mathbf{p}$ has height r. Therefore every saturated chain of prime ideals from $\mathbf{p}$ to $\mathbf{m}$ has length $n-r$. It follows that $\mathbf{m}/I'$ has height $n-r$ in B'.

Let $(C^{(2)}_{\cdot}, \delta_2)$ be the Koszul complex on $K[x]$ defined by $f_1, \dots, f_r$. We regard $C^{(2)}_q$ as the free $K[x]$-module with basis

$$\zeta_{k_1} \cdots \zeta_{k_q},$$

where $1 \leq k_1 < \cdots < k_q \leq r$, and $\delta_2 : C^{(2)}_q \to C^{(2)}_{q-1}$ is defined by

$$\delta_2(\zeta_{k_1} \cdots \zeta_{k_q}) = \sum_{l=1}^{q} (-1)^{l-1} f_{k_l}\, \zeta_{k_1} \cdots \hat{\zeta}_{k_l} \cdots \zeta_{k_q}.$$

Each $C^{(2)}_q$ is graded by using the usual grading on $K[x]$ and by defining the degree of $\zeta_{k_1} \cdots \zeta_{k_q}$ to be $d_{k_1} + \cdots + d_{k_q}$. The complex $(C^{(2)}_{\cdot}, \delta_2)$ becomes a filtered complex by defining level k of the filtration to be the K-span of homogeneous elements of degree $\leq k$. Its associated graded complex $(\bar{C}^{(2)}_{\cdot}, \bar{\delta}_2)$ is the Koszul complex on $K[x]$ defined by $f_1^{(d_1)}, \dots, f_r^{(d_r)}$.

Consider the double complex $C^{(1)}_{\cdot} \bigotimes_{K[x]} C^{(2)}_{\cdot}$ and let $T_{\cdot}$ be its associated total complex. The filtrations on $C^{(1)}_{\cdot}$ and $C^{(2)}_{\cdot}$ make $T_{\cdot}$ a filtered complex. Its associated graded complex $\bar{T}_{\cdot}$ is the total complex of the double complex $\bar{C}^{(1)}_{\cdot} \bigotimes_{K[x]} \bar{C}^{(2)}_{\cdot}$. One checks easily from the definitions that

$$H_0(T_{\cdot}) = K[x]/(I+J), \tag{5.4}$$

$$H_0(\bar{T}_{\cdot}) = K[x]/(I'+J'). \tag{5.5}$$

By (5.4) and Corollary 1.4,

$$H_0(T_{\cdot}) \simeq H^{n-r}(\Omega^{\cdot}_{B/K}, \phi_{\bar{f}}). \tag{5.6}$$

We determine the relation between $H_0(T_{\cdot})$ and $H_0(\bar{T}_{\cdot})$.

Since $f_1^{(d_1)}, \dots, f_r^{(d_r)}$ is a regular sequence in $K[x]$,

$$H_p(\bar{C}^{(2)}_{\cdot}) = 0 \qquad \text{for } p > 0.$$

Furthermore, $H_0(\bar{C}^{(2)}_{\cdot}) \simeq B'$. This implies by standard homological algebra that

$$H_p(\bar{T}.) \simeq H_p\biggl(\bar{C}^{(1)}_{\cdot} \bigotimes_{K[x]} B'\biggr),$$

hence by Proposition 5.3

$$H_p(\bar{T}.) = 0 \qquad \text{for } p > 0. \tag{5.7}$$

Standard homological algebra then implies that

$$H_p(T.) = 0 \qquad \text{for } p > 0$$

and that

$$\mathrm{gr}(H_0(T.)) \simeq H_0(\bar{T}.) \tag{5.8}$$

as K-vector spaces, where the left-hand side denotes the associated graded relative to the filtration induced by $T.$ on $H_0(T.)$. In particular, (5.6) implies that

$$\dim_K H^{n-r}(\Omega^{\cdot}_{B/K}, \phi_{\bar{f}}) = \dim_K H_0(\bar{T}.) \tag{5.9}$$

Proposition 5.2 and standard homological algebra imply that

$$H_p(\bar{T}.) \simeq H_p\biggl(H_0(\bar{C}^{(1)}_{\cdot}) \bigotimes_{K[x]} \bar{C}^{(2)}_{\cdot}\biggr). \tag{5.10}$$

Thus by (5.7), $H_0(\bar{C}^{(1)}_{\cdot}) \bigotimes_{K[x]} \bar{C}^{(2)}_{\cdot}$ is a resolution of

$$\begin{aligned} H_0\biggl(H_0(\bar{C}^{(1)}_{\cdot}) \bigotimes_{K[x]} \bar{C}^{(2)}_{\cdot}\biggr) &\simeq H_0(\bar{C}^{(1)}_{\cdot}) \bigotimes_{K[x]} H_0(\bar{C}^{(2)}_{\cdot}) \\ &\simeq H_0(\bar{C}^{(1)}_{\cdot}) \bigotimes_{K[x]} B'. \end{aligned}$$

But $H_0(\bar{C}^{(1)}_{\cdot}) \bigotimes_{K[x]} \bar{C}^{(2)}_{\cdot}$ is just the Koszul complex on $H_0(\bar{C}^{(1)}_{\cdot})$ defined by $f_1^{(d_1)}$, $\dots$, $f_r^{(d_r)}$. It then follows from Proposition 5.2 that the Hilbert-Poincaré series of $H_0(\bar{C}^{(1)}_{\cdot}) \bigotimes_{K[x]} B'$ is

$$\biggl(\prod_{i=1}^{r}(1-t^{d_i})\biggr)\frac{G(t)(1-t)^{n-r} + H(t)(1-t)^{n-r+1}}{(1-t)^n}. \tag{5.11}$$

By (5.7) and (5.10), this is the Hilbert-Poincaré series of $H_0(\bar{T}.)$, hence

$$\dim_K H_0(\bar{T}.) = d_1 \cdots d_r G(1).$$

By Proposition 5.2, this proves that $\dim_K H_0(\bar{T}.)$ equals the coefficient of t^{n-r} in the power series expansion at $t = 0$ of

$$\frac{d_1 \cdots d_r (1-t)^n}{\prod_{i=1}^{r+1}(1-d_i t)}.$$

Proposition 1.5 then follows from (5.9).

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